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EXPONENTIAL WEAK BERNOULLI MIXING FOR COLLET–ECKMANN MAPS

BY

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ABSTRACT
We prove exponential weak Bernoulli mixing for invariant measures of certain piecewise monotone interval maps studied in [BK] and [KN]. In particular we prove this for unimodal maps with negative Schwarzian derivative satisfying $\liminf_{n\to\infty} \sqrt[n]{|DT^n(Tc)|} > 1$, where c is the unique critical point of T.

1. Introduction

The aim of this note is to prove exponential weak Bernoulli mixing for invariant measures of certain interval maps studied in [BK] and [KN]. The main result concerns Collet–Eckmann maps $T\colon [0,1] \to [0,1]$. These are unimodal maps of class C^3 with negative Schwarzian derivative

$$ST := \frac{T'''}{T'} - \frac{3}{2}\left(\frac{T''}{T'}\right)^2 \le 0 \qquad \text{except at } c \text{ where } T' = 0.$$

For such maps Collet and Eckmann [CE] proved: If $\liminf_{n\to\infty} \sqrt[n]{|DT^n(Tc)|} > 1$, then T has an invariant probability density. (Indeed, they used some additional assumption, which was removed in [No].) We call this class of maps **Collet–Eckmann maps.**

As a consequence of the general metric theory of S-unimodal maps [BL, Ke, Le] it is known that an invariant probability density h, if it exists at all, gives rise to a measure preserving dynamical system which is mixing (and even weakly Bernoulli) up to a finite rotation. This means that there is a finite disjoint

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collection of p intervals $I_0, \dots, I_{p-1}$ which are cyclically permuted by T, and T^p, restricted to any of these intervals, is unimodal and mixing. If $p = 1$, T is called **nonrenormalizable**, otherwise we say T is **finitely renormalizable**.

In particular, if T is nonrenormalizable, the natural partition of [0,1] into the monotonicity intervals $(0, c)$ and $(c, 1)$ of T is a weak Bernoulli generator. This means that the σ-algebra $\mathcal{F}_0^\infty$ coincides, up to sets of Lebesgue measure 0, with the Borel σ-algebra and that

$$\beta_n(T, \mathcal{Z}, \mu) \to 0 \quad \text{as } n \to \infty,$$

where

$$\beta_n(T, \mathcal{Z}, \mu) := 2 \cdot \sup_{k>0} \int \sup\{|\mu(A|\mathcal{F}_0^k) - \mu(A)| \colon A \in \mathcal{F}_{k+n}^\infty\}\, d\mu, \tag{1.1}$$

$\mu := h \cdot m$ denotes the invariant measure, and $\mathcal{F}_k^\ell$ is the σ-algebra generated by the partitions $T^{-i}\mathcal{Z}$ $(k \le i < \ell)$.

The aim of this note is to prove

THEOREM 1: *If T is a nonrenormalizable Collet–Eckmann map satisfying the additional regularity assumptions* (1.2) *below, then there are $C > 0$ and $0 < r < 1$ such that $\beta_n(T, \mathcal{Z}, \mu) \le C \cdot r^n$ for all $n > 0$.*

The weaker assertion that correlations of sufficiently well behaved functions decrease exponentially to 0 was already proved in [KN, Theorem 1.1] and [Yo].

The additional regularity assumptions are

There is a constant $M > 0$ such that

$$\begin{aligned}
&1.\quad M^{-1} < \frac{|x-c|^{\ell-1}}{|DT(x)|} < M \quad \text{for all } x,\\
&2.\quad \operatorname{var}_{[0,1]} \frac{|x-c|^{\ell-1}}{|DT(x)|} < M \quad \text{and}\\
&3.\quad \operatorname{var}_{[0,u]} \frac{|Tx-Tu|}{|x-u||DT(x)|} \quad \text{and} \quad \operatorname{var}_{[v,1]} \frac{|Tx-Tv|}{|x-v||DT(x)|} < M\\
&\quad\ \ \text{if } u < c < v.
\end{aligned} \tag{1.2}$$

These conditions are satisfied e.g. if T is a polynomial map with vanishing derivatives at c of all orders up to $\ell - 1$, but also for $T(x) = a(1 - |2x-1|^\ell)$ with real $\ell > 1$. In both cases conditions 1. and 2. are easily checked. For condition 3. one should observe that the expressions of interest are bounded by 1 if both, x

and u (resp. v), are close to c, and that the derivatives of these expressions have a bounded number of sign changes.

As in [KN] we reduce the investigation of Collet–Eckmann maps to the situation studied in [BK].

Theorem 1 is useful for proving probabilistic limit theorems like the central limit theorem or the law of the iterated logarithm for statistics based on samples $(x, Tx, \ldots, T^{n-1}x)$ where $x \in [0,1]$ is picked at random from $([0,1], \text{Lebesgue})$ or $([0,1], \mu)$, see e.g. [HK, Theorem 5] and [DK]. A central limit theorem for the process $(x, Tx, T^2x, \ldots)$ itself is already proved in [KN,Yo].

2. Maps with countably many monotone branches

In this section let $\mathcal{I}$ be a finite or countable family of intervals (which are subsets of some totally ordered, order complete space, cf. [HK, Ry]). Let X be the disjoint union of these intervals $I \in \mathcal{I}$. Suppose further there is a family $\mathcal{Z}$ of disjoint subintervals of X (in particular, for each $Z \in \mathcal{Z}$ there is $I \in \mathcal{I}$ with $Z \subseteq I$) and let $Y = \bigcup_{Z \in \mathcal{Z}} Z$.

We study a transformation $T: Y \to X$ such that for each $Z \in \mathcal{Z}$ holds

$$\exists\, I \in \mathcal{I} \text{ such that } T(Z) \subseteq I \quad \text{and} \tag{2.1}$$

$$\begin{gathered} T_{|Z} \text{ is monotone on } Z \text{ and has the Darboux property} \\ \text{(i.e., if } J \subseteq I \text{ is an interval, then } TJ \text{ is an interval).} \end{gathered} \tag{2.2}$$

For an interval $J \subseteq X$ and a function $f: X \to \mathbf{C}$ define

$$\begin{aligned} \operatorname{var}_J(f) &:= \sup\Big\{ \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})| : \\ &\qquad\qquad n \geq 1, a_0 < a_1 < \cdots < a_n, a_i \in J \Big\}, \\ \operatorname{var}(f) &:= \sum_{I \in \mathcal{I}} \operatorname{var}_I(f), \\ \|f\|_\infty &:= \sum_{I \in \mathcal{I}} \sup_I |f|, \\ \|f\|_{BV} &:= \operatorname{var}(f) + \|f\|_\infty. \end{aligned}$$

Let $BV := \{f: X \to \mathbf{C}: \|f\|_{BV} < \infty\}$. Then $(BV, \|\,.\,\|_{BV})$ is a Banach space. Fix $g: X \to \mathbf{C}$ and define $g_n: X \to \mathbf{C}$ by

$$g_n(x) := g(x) \cdot g(Tx) \cdots g(T^{n-1}x)$$

and

$$\vartheta := \lim_{n\to\infty} \left(\sup_X |g_n| \right)^{1/n} . \tag{2.3}$$

We make the following assumption on g:

$$M_1 := \sup \left\{ \operatorname{var}_I(g) + \sum_{\substack{Z\in\mathcal{Z}\\ Z\subseteq I}} \sup_Z |g| \colon I \in \mathcal{I} \right\} < \infty. \tag{2.4}$$

Associated with T and g is the transfer operator

$$\mathcal{L}\colon BV \to BV, \quad f \mapsto \sum_{Z\in\mathcal{Z}} (f\cdot g)\circ T_{|Z}^{-1}.$$

If $\mathcal{Z}_n := \{Z_0 \cap T^{-1}Z_1 \cap \cdots \cap T^{-(n-1)}Z_{n-1} \colon Z_i \in \mathcal{Z} \text{ for all } i\}$ and $T_\eta^{-n} := (T^n{}_{|\eta})^{-1}$ for $\eta \in \mathcal{Z}_n$, then

$$\mathcal{L}^n f = \sum_{\eta\in\mathcal{Z}_n} (f\cdot g_n)\circ T_\eta^{-n}.$$

By $|\mathcal{L}|$ we denote the transfer operator associated with T and $|g|$. Both, $\mathcal{L}$ and $|\mathcal{L}|$ are bounded linear operators on BV, see [BK, Lemma 2.2].

LEMMA 1: *Let $J \subseteq I \in \mathcal{I}$ be an interval, $f_1, f_2\colon J \to \mathbf{C}$. Then*

$$\|\chi_J \cdot f_1 \cdot f_2\|_{BV} \le \left(\operatorname{var}_J(f_1) + 5\sup_J |f_1| \right) \left(\operatorname{var}_J(f_2) + \inf_J |f_2| \right).$$

Proof:

$$\begin{aligned} \|\chi_J \cdot f_1 \cdot f_2\|_{BV} &\le \sup_J |f_1| \cdot \operatorname{var}(\chi_J \cdot f_2) + \left(\operatorname{var}(\chi_J\cdot f_1) + \sup_J |f_1| \right) \cdot \sup_J |f_2| \\ &\le \sup_J |f_1| \cdot 2\left(\operatorname{var}_J(f_2) + \inf_J |f_2| \right) \\ &\qquad + \left(\operatorname{var}_J(f_1) + 3\sup_J |f_1| \right) \left(\operatorname{var}_J(f_2) + \inf_J |f_2| \right) \\ &\le \left(\operatorname{var}_J(f_1) + 5\sup_J |f_1| \right) \left(\operatorname{var}_J(f_2) + \inf_J |f_2| \right) \qquad \blacksquare \end{aligned}$$

The following lemma is an adaption of an estimate from [Ry] to our setting.

LEMMA 2: *For $f \in BV$ let*

$$v_0(f) := \|f\|_{BV}, \quad v_n(f) := \sum_{\eta \in \mathcal{Z}_n} \|\mathcal{L}^n(\chi_\eta \cdot f)\|_{BV} \quad (n > 0).$$

For each $\Theta > \vartheta$ there is a constant $C > 0$ independent of f such that for all $n \geq 0$

$$v_n(f) \leq C \cdot \left(\Theta^n \|f\|_{BV} + \sum_{k=0}^{n-1} \Theta^k \| |\mathcal{L}|^{n-k} |f| \|_\infty \right).$$

Proof: Consider $\eta = \eta' \cap T^{-k}\eta''$ where $\eta' \in \mathcal{Z}_k$, $\eta'' \in \mathcal{Z}_\ell$, and $n = k + \ell$. Then, writing $J := T^n\eta$, $J' := T^k\eta$, $\varphi := (g_\ell \cdot \chi_{\eta''}) \circ T_{\eta''}^{-\ell} = \mathcal{L}^\ell \chi_{\eta''}$ and $\psi := \mathcal{L}^k(\chi_{\eta'} \cdot f)$, we have

$$\begin{aligned}
\|\mathcal{L}^{k+\ell}(\chi_\eta \cdot f)\|_{BV} &= \|\chi_J \cdot \mathcal{L}^\ell(\chi_{\eta''} \cdot \mathcal{L}^k(\chi_{\eta'} \cdot f))\|_{BV} \\
&= \|\chi_J \cdot \varphi \cdot (\psi \circ T_{\eta''}^{-\ell})\|_{BV} \\
&\leq \left(\operatorname{var}_J(\varphi) + 5 \sup_J |\varphi| \right) \left(\operatorname{var}_{J'}(\psi) + \inf_{J'} |\psi| \right) \qquad \text{Lemma 1} \\
&\leq \left(\operatorname{var}_{\eta''}(g_\ell) + 5 \sup_{\eta''} |g_\ell| \right) \cdot \operatorname{var}_{\eta''}(\psi) + 5 \|\mathcal{L}^\ell \chi_{\eta''}\|_{BV} \inf_{\eta''} |\psi| .
\end{aligned}$$

As $J = T^n\eta \subseteq I$ for some $I \in \mathcal{I}$ and as $\operatorname{var}_{\eta''}(g_\ell) + 5 \sup_{\eta''} |g_\ell| \leq C_1 \cdot \Theta^\ell$ with some constant $C_1 > 0$ by [BK, Corollary 2.4], this yields

$$\begin{aligned}
v_{k+\ell}(f) &= \sum_\eta \|\mathcal{L}^{k+\ell}(\chi_\eta \cdot f)\|_{BV} \\
&\leq C_1 \cdot \Theta^\ell \cdot \sum_{\eta'} \operatorname{var}(\mathcal{L}^k(\chi_{\eta'} \cdot f)) \qquad (2.5) \\
&\quad + 5 \sum_{I \in \mathcal{I}} \sum_{\eta'' \subseteq I} \|\mathcal{L}^\ell \chi_{\eta''}\|_{BV} \cdot \sum_{\eta'} \inf_{\eta''} (|\mathcal{L}|^k (\chi_{\eta'} \cdot |f|)) \\
&\leq C_1 \cdot \Theta^\ell \cdot v_k(f) + 5 \sum_{I \in \mathcal{I}} \sum_{\eta'' \subseteq I} \|\mathcal{L}^\ell \chi_{\eta''}\|_{BV} \cdot \inf_{\eta''} (|\mathcal{L}|^k |f|) \\
&\leq C_1 \cdot \Theta^\ell \cdot v_k(f) + 5 \sum_{I \in \mathcal{I}} v_\ell(\chi_I) \cdot \sup_I |\mathcal{L}|^k |f| . \qquad (2.6)
\end{aligned}$$

Observe next that for each $I \in \mathcal{I}$

$$v_1(\chi_I) \leq \sum_{\substack{Z \in \mathcal{Z} \\ Z \subseteq I}} \|\mathcal{L}\chi_Z\|_{BV} \leq \sum_{\substack{Z \in \mathcal{Z} \\ Z \subseteq I}} \left(\operatorname{var}_Z(g) + 2 \sup_Z |g| \right) \leq 2M_1$$

by (2.4). Hence (2.5) yields, if applied to $f = \chi_{I_0}$, $I_0 \in \mathcal{I}$, and $\ell = 1$

$$v_{k+1}(\chi_{I_0}) \leq C_1\Theta \cdot v_k(\chi_{I_0}) + 10M_1 \cdot \| |\mathcal{L}|^k \chi_{I_0} \|_\infty ,$$

and it follows by induction that

$$S_k := \sup\{v_k(\chi_I): I \in \mathcal{I}\} < \infty$$

for all $k > 0$. Therefore we can continue (2.5) by

$$v_{k+\ell}(f) \leq C_1 \cdot \Theta^\ell \cdot v_k(f) + 10S_\ell \cdot \| |\mathcal{L}|^k |f| \|_\infty . \tag{2.7}$$

Now pick $\bar{\Theta} \in (\vartheta, \Theta)$ and fix $\ell_0 > 0$ such that $C_1 \cdot \bar{\Theta}^{\ell_0} < \Theta^{\ell_0}$. Then (2.7), for $\bar{\Theta}$ instead of Θ, results in

$$v_{k+\ell_0}(f) \leq \Theta^{\ell_0} \cdot v_k(f) + 10S_{\ell_0} \cdot \| |\mathcal{L}|^k |f| \|_\infty ,$$

and induction yields for all $n = (j+1)\ell_0$ $(j \geq 0)$

$$v_{(j+1)\ell_0}(f) \leq \Theta^{(j+1)\ell_0} \|f\|_{BV} + 10S_{\ell_0} \cdot \sum_{i=0}^{j} \Theta^{i\ell_0} \| |\mathcal{L}|^{(j-i)\ell_0} |f| \|_\infty .$$

Another application of (2.7) gives the desired estimate for arbitrary n. ∎

Suppose now that there is a Borel measure m on X such that

$$\frac{d(m \circ T)}{dm} = \frac{1}{g}, \quad \vartheta < 1, \quad \text{and} \quad M_2 := \sup\{m(I): I \in \mathcal{I}\} < \infty. \tag{2.8}$$

(In particular, $g \geq 0$ and $|\mathcal{L}| = \mathcal{L}$.) Then

$$\int \mathcal{L}f \, dm = \int f \, dm \quad \text{for all } f \in L^1_m \tag{2.9}$$

and

$$\int |f| \, dm \leq \sum_{I \in \mathcal{I}} m(I) \cdot \sup_I |f| \leq M_2 \cdot \|f\|_\infty < \infty \tag{2.10}$$

for all $f \in BV$. Under some additional assumptions, which will be discussed below, it is shown in [BK] (cf. also [Ry]) that $\mathcal{L}$ has the following spectral decomposition:

$$\mathcal{L} = \sum_{i=1}^{N} \lambda_i \mathcal{P}_i + \mathcal{Q} , \tag{2.11}$$

where $\lambda_1 = 1$, $|\lambda_i| = 1$ $(i = 1, \ldots, N)$, the $\mathcal{P}_i$ are finite rank projections, $\mathcal{P}_i\mathcal{P}_j = \mathcal{P}_i\mathcal{Q} = 0$ $(i \neq j)$, and there are $K > 0$ and $r \in (\vartheta, 1)$ such that $\|\mathcal{Q}^n\|_{BV} \leq K \cdot r^n$ $(n > 0)$.

Note: The facts that the spectral radius $\rho(\mathcal{L}) = 1$ and that the leading eigenvalues of $\mathcal{L}$ are semisimple are deduced from (2.9).

Each of the following additional assumptions is sufficient for the spectral decomposition (2.11):

(A) $\mathrm{var}\,(g) + \sum_{Z\in\mathcal{Z}} \sup_Z |g| < \infty$. (See [Ry] or [BK, Theorem 2.8].)

(B) (X, T) is the Markov extension of a piecewise monotonic interval map with finitely many monotone branches, and g and m are the lifts of the corresponding objects of the interval map to X, see (4.1) in [BK].

(C) (X, T) is the Markov extension of a Collet–Eckmann map, and g and m are in a suitable way multiplicatively cohomologous to the lifts of the function $1/|T'|$ and the Lebesgue measure on $[0, 1]$ respectively, see [KN].

The two last cases will be discussed in more detail in the next section.

We shall say that $\mathcal{L}$ is **mixing,** if $\lambda_1 = 1$ is the only eigenvalue of modulus 1 and if it is a simple eigenvalue of $\mathcal{L}$.

THEOREM 2: *Suppose (X, T), g and m are as described above (in particular $\vartheta < \infty$) and such that the spectral decomposition* (2.11) *holds. If $\mathcal{L}$ is mixing, then there is a unique $h \in BV$ with $h \geq 0$, $\int h\,dm = 1$, $\mathcal{L}h = h$, and for the T-invariant measure $\mu = h \cdot m$*

$$\beta_n(T, \mathcal{Z}, \mu) \leq \text{const} \cdot r^n$$

holds with $r < 1$ from (2.11). *($\beta_n(T, \mathcal{Z}, \mu)$ was defined in* (1.1).*)*

Proof (Cf. [Ry]): Consider $A \in \mathcal{F}_{n+k}^{\infty}$, $A = T^{-(n+k)}\tilde{A}$ with $\tilde{A} \in \mathcal{F}_0^{\infty}$, and $\eta \in \mathcal{Z}_k$. We have

$$\mu(A|\mathcal{Z}_k)_{|\eta} = \frac{1}{\mu(\eta)} \int_A \chi_\eta \cdot h\,dm = \frac{1}{\mu(\eta)} \int_{\tilde{A}} \mathcal{L}^{n+k}(\chi_\eta \cdot h)\,dm\ ,$$

and as $\mu(A) = \mu(\tilde{A})$ and $\mathcal{P}_1(\mathcal{L}^k(\chi_\eta \cdot h)) = \int_\eta h\,dm \cdot h = \mu(\eta) \cdot h$ by (2.8), it follows that

$$\begin{aligned}
|\mu(A|\mathcal{Z}_k) - \mu(A)|_{|\eta} &= \left| \int_{\tilde{A}} \left(\frac{1}{\mu(\eta)} \mathcal{L}^{n+k}(\chi_\eta \cdot h) - h \right) dm \right| \\
&\leq \frac{1}{\mu(\eta)} \int \left| \mathcal{L}^{n+k}(\chi_\eta \cdot h) - \mu(\eta) \cdot h \right| dm \\
&= \frac{1}{\mu(\eta)} \int \left| \mathcal{Q}^n(\mathcal{L}^k(\chi_\eta \cdot h)) \right| dm\ .
\end{aligned}$$

This implies

$$\begin{aligned}\beta_n(T,\mathcal{Z},\mu) &= 2\cdot\sup_{k>0}\int \sup\{|\mu(A|\mathcal{F}_0^k)-\mu(A)|\colon A\in\mathcal{F}_{k+n}^{\infty}\}\,d\mu\\ &\le 2\cdot\sup_{k>0}\left(\sum_{\eta\in\mathcal{Z}_k} M_2\cdot\|\mathcal{Q}^n\mathcal{L}^k(\chi_\eta\cdot h)\|_\infty\right)\quad\text{by (2.10)}\\ &\le 2M_2\cdot\|\mathcal{Q}^n\|_{BV}\cdot\sup_{k>0}\left(\sum_{\eta\in\mathcal{Z}_k}\|\mathcal{L}^k(\chi_\eta\cdot h)\|_\infty\right)\\ &\le 2M_2\cdot Kr^n\cdot\sup_{k>0} v_k(h)\\ &\le 2M_2\cdot Kr^n\cdot\frac{C}{1-\Theta}\end{aligned}$$

by (2.11) and Lemma 2 for some $C>0$ and $\Theta\in(\vartheta,1)$. ∎

3. Proof of Theorem 1

In this section X is denotes a linearly ordered, order complete set (which is compact as a topological space with its order topology), and $T\colon X\to X$ is supposed to be a piecewise monotonic transformation, i.e. there is a finite partition $\mathcal{Z}$ of X into intervals such that $T_{|Z}$ is monotone and continuous for each $Z\in\mathcal{Z}$. Let also a Borel probability measure m on X and a function $g\colon X\to\mathbf{R}$ of bounded variation be given such that $1/g=d(m\circ T)/dm$.

The Markov extension of (X,T) is a dynamical system $(\widehat{X},\widehat{T})$ together with a projection $\pi\colon\widehat{X}\to X$ such that $T\circ\pi=\pi\circ\widehat{T}$. Its phase space $\widehat{X}=\bigcup_{I\in\mathcal{I}} I$ is a disjoint union of at most countably many subintervals I of X, and $\pi_{|I}$ (for $I\in\mathcal{I}$) is just the canonical embedding of I into X. This construction, which is due to Hofbauer, is described in [BK], where also further references are given.

Let $\hat{\mathcal{Z}}:=\mathcal{I}\vee\pi^{-1}\mathcal{Z}$, $\hat{g}:=g\circ\pi$. If we define the Borel measure $\widehat{m}$ on $\widehat{X}$ by $\widehat{m}(A)=m(\pi(A))$ for measurable $A\subseteq I\in\mathcal{I}$, then the system $(\widehat{X},\widehat{T})$ together with the partitions $\mathcal{I}$ and $\widehat{\mathcal{Z}}$, the weight function $\hat{g}$ and the measure $\widehat{m}$ fits into the setting of section 2, see section 3 and (4.1) of [BK]. Let us denote by $\widehat{BV}$, $\widehat{\mathcal{L}}$, etc. the objects BV, $\mathcal{L}$, etc. for $\widehat{X}$, and observe that $\hat{\vartheta}=\vartheta$. If $\vartheta<1$, then $\widehat{\mathcal{L}}$ has the spectral decomposition (2.11), and if $\widehat{\mathcal{L}}$ is mixing, then there is a unique $\hat{h}\in\widehat{BV}$ such that $\hat{\mu}=\hat{h}\cdot\widehat{m}$ is a $\widehat{T}$-invariant probability measure, and by Theorem 2, $\beta_n(\widehat{T},\widehat{\mathcal{Z}},\hat{\mu})\le\text{const}\cdot r^n$ for some $r\in(\vartheta,1)$. Let $\mu:=\hat{\mu}\circ\pi^{-1}$. Obviously, $\beta_n(T,\mathcal{Z},\mu)\le\beta_n(\widehat{T},\widehat{\mathcal{Z}},\hat{\mu})$, and we have proved

COROLLARY 1 ([HK, Ry]): *If (X,T), $\mathcal{Z}$, g, m and μ are as above and if $\vartheta < 1$, then there is $r \in (\vartheta, 1)$ such that $\beta_n(T, \mathcal{Z}, \mu) \leq \text{const} \cdot r^n$.*

Suppose now that T: $[0,1] \to [0,1]$ is a nonrenormalizable Collet–Eckmann map (see section 1). With $g = 1/|T'|$ and m = Lebesgue measure we are in the situation described above except that g is unbounded (as $T'(c) = 0$) such that neither is g of bounded variation nor is $\vartheta < \infty$. In order to overcome this difficulty a function $\widehat{w}$: $\widehat{X} \to (0,\infty)$ was constructed in [KN] such that the weight function

$$\hat{g}: \widehat{X} \to \mathbf{R}, \quad \hat{g} := (g \circ \pi) \cdot \frac{\widehat{w}}{\widehat{w} \circ \widehat{T}}$$

satisfies 2.4 [KN, Prop. 6.2] and $\hat{\vartheta} < 1$ [KN, Prop. 6.3]. If $\widehat{m}$ denotes the measure on $\widehat{X}$ with density $\widehat{w}$ with respect to the Lebesgue measure on $\widehat{X}$, then obviously $1/\hat{g} = d(\widehat{m} \circ \widehat{T})/d\widehat{m}$ and $\sup\{\widehat{m}(I): I \in \mathcal{I}\} < \infty$ [KN, Prop. 6.1], i.e. assumption (2.8) is satisfied. Furthermore, Theorem 2.1 of [KN] and the discussion at the beginning of section 5 of that paper show that $\widehat{\mathcal{L}}$ has the spectral decomposition (2.11) and is mixing. In particular, there is a unique $\hat{h} \in \widehat{BV}$ with $0 \leq \hat{h} = \widehat{\mathcal{L}}\hat{h}$, and the system $(\widehat{T}, \hat{\mu} := \hat{h}\widehat{m})$ is mixing. Hence Theorem 2 applies to $(\widehat{X}, \widehat{T})$, $\hat{g}$, $\widehat{m}$ and $\hat{\mu}$. Denoting by $\mu := \hat{\mu} \circ \pi^{-1}$ the projection of $\hat{\mu}$ to X, we conclude

$$\beta_n(T, \mathcal{Z}, \mu) \leq \beta_n(\widehat{T}, \hat{\mathcal{Z}}, \hat{\mu}) \leq \text{const} \cdot r^n \quad \text{for some } r \in (\hat{\vartheta}, 1) .$$

This proves Theorem 1 from the introduction.

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